

Teaching superfluidity at the introductory level

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Standard introductory modern physics textbooks do not exactly dwell on superfluidity in ^4He . Typically, Bose-Einstein condensation (BEC) is mentioned in the context of an ideal Bose gas, followed by the statement that BEC happens in ^4He and that the ground state of ^4He exhibits many interesting properties such as having zero viscosity. Not only does this approach not explain in any way why ^4He becomes a superfluid, it denies students the opportunity to learn about the far reaching consequences of energy gaps as they develop in both superfluids and superconductors. We revisit superfluid ^4He by starting with Feynman's explanation of superfluidity based on Bose statistics as opposed to BEC, and we present exercises for the students that allow them to arrive at a very accurate estimate of the superfluid transition temperature and of the energy gap separating the ground state from the first excited state. This paper represents a self-contained account of superfluidity, which can be covered in one or two lessons in class.

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INTRODUCTION

Superfluidity is the property of a liquid to flow without friction through thin capillaries [1]. This property is manifest in ^4He below $T_\lambda = 2.17$ K, the so called superfluid or lambda-transition (named after the shape of the specific heat curve). Below 1 K, 100% of the liquid exhibits this property. Bose-Einstein condensation (BEC) on the other hand, is the property that a large fraction of the particles that make up a system condense into the same state. For instance, in an ideal Bose gas (a gas made up of bosons that do not interact with each other), 100% of the particles will condense into the state with the lowest available energy and form a Bose-Einstein condensate. However, an ideal Bose gas does not become superfluid. And conversely, in liquid ^4He only about 7% [2] of the atoms actually form a condensate, even though essentially 100% of the atoms can flow without friction below 1 K.

In fact, there is no reason why a system could not become a superfluid even if only a very small fraction of the atoms were to form a condensate. All this nicely illustrates the fact that superfluidity and BEC are two different phenomena, even though introductory textbooks tend to lump the two together. The main difference between BEC and superfluidity is that BEC is a property of the ground state, while superfluidity is a property of the excited states. This is entirely analogous to standard superconductivity, where the electrons condense into Cooper pairs (ground state), and where the interaction between the Cooper pairs introduces a finite energy gap between the ground state and excited states. In turn, this energy gap is responsible for the system becoming a superconductor. Thus, in both systems, it is the interaction between the particles that is responsible for the exotic behaviors, not how they

arrange themselves in the ground state.

In this paper we focus on the property of superfluidity rather than on BEC. We repeat Feynman's arguments that show that any Bose liquid that stays liquid down to low enough temperatures must become a superfluid because of the presence of an energy gap. We also derive a very accurate estimate of the superfluid transition temperature using basic conservation laws and some straightforward approximations. Altogether, this should give students a much better understanding of what superfluidity entails and why it necessarily must occur in ^4He . In addition, our simple calculations should bestow upon them the idea that they have already learned enough physics to be able to come up with a very accurate estimate of something as complex as the superfluid transition temperature in ^4He . Note that we do not argue that a Bose condensate does not form in ^4He , rather we focus on the majority of non-condensed atoms (93 %) that determine the numerical values for virtually all quantities of interest.

SUPERFLUIDITY: QUALITATIVE UNDERSTANDING

First, the fact that helium does not solidify at any temperature is a pure quantum effect. The weak van der Waals forces between the atoms are not strong enough to overcome the zero point motion associated with trying to confine a helium atom to a lattice site. The second aspect that makes helium stand out from other liquids is that it takes a finite amount of energy to create a disturbance in the liquid. This is shown in Fig. 1. The actual amount of energy required depends on the wavelength λ (or momentum $p = h/\lambda$) of this disturbance. The mea-

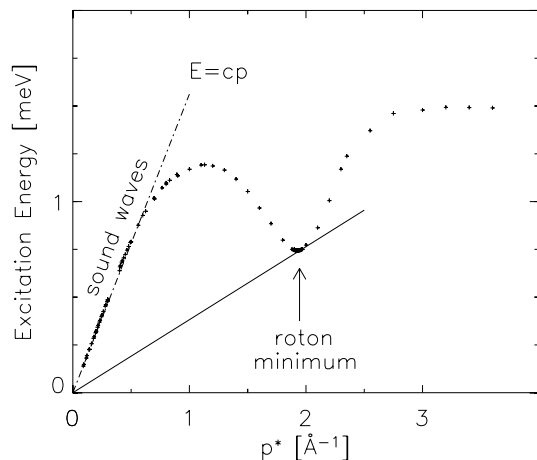


FIG. 1: The measured excitation energies $E(p)$ [3] in superfluid ^4He as a function of momentum transfer $p^* = p/\hbar = 2\pi/\lambda$. The slope of the curve at small p^* [dashed line] is given by the velocity of sound $c = 237.4$ m/s [16], the overall minimum slope [corresponding to a speed of 58 m/s] is given by the solid line which is tangent to the excitation curve near the so-called roton minimum [$p^* = 1.94 \text{ \AA}^{-1}$, $E(p) = 0.743$ meV].

sured values [3] for the energy cost are shown in Fig. 1. At low momentum transfers (long wave lengths) the energy disturbance is just a run-of-the-mill sound wave (a phonon), and its energy is given by $E_{ex}(p) = cp$, the standard hydrodynamics result for any liquid, not just superfluids [4].

When we go to lower wave lengths, such as the density disturbance pictured in Fig. 2, the energy cost starts to deviate from $E_{ex}(p) = cp$. For wave lengths comparable to the interatomic spacing d , the energy cost goes through a minimum, after which it goes up again. This minimum of the energy gap between the ground state and the excited state is commonly referred to in the literature on superfluid helium as the roton minimum [5] or simply 'the roton', and the entire curve is referred to as the phonon-roton dispersion curve. The roton turns out to be the determining feature of superfluids. As pointed out in the preceding, the presence of this roton gap is analogous to the presence of a similar gap in superconducting systems. The presence of a gap also firmly sets superfluid ^4He apart from normal fluids where nothing resembling an energy gap exists. This is shown in Fig. 3 where we compare helium in the superfluid phase to helium in the normal fluid phase.

One can easily verify from Fig. 1 that the presence of a non-zero energy gap is synonymous with superfluidity. The slope of a line that goes through the origin and a point on the excitation curve gives the (group) velocity of the excitation. For instance, this slope at small momenta is given by the speed of sound [see Fig. 1]. The overall smallest slope is encountered near the roton minimum. The value of the slope at this point corresponds

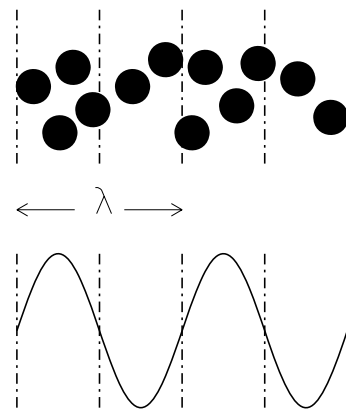


FIG. 2: A real space visualization of a density disturbance [a departure from the average density of 3 in this figure] that resembles a sound wave with a wave length λ of about 3 times the average atomic separation. The roton minimum corresponds to a disturbance with a wave length comparable to the atomic separation, but there is no agreement on how to visualize such a short wave length excitation [5]

to the velocity below which the liquid can flow without friction, at least through small capillaries [6]. After all, if the liquid is flowing at a lower speed, then the liquid cannot slow down because of the following restrictions due to the energy and momentum conservation laws. Following standard arguments [7], we focus on a liquid mass M that is flowing at speed v . For it to slow down to speed v' by creating an excitation of energy E_{ex} and momentum \vec{p}_{ex} we have

$$\begin{aligned} Mv^2/2 &= Mv'^2/2 + E_{ex} \\ M\vec{v} &= M\vec{v}' + \vec{p}_{ex}. \end{aligned} \quad (1)$$

Eliminating v' we get

$$\vec{v} \cdot \vec{p}_{ex} - p_{ex}^2/2M = E_{ex}. \quad (2)$$

Even in the best case scenario in which \vec{v} and \vec{p}_{ex} are parallel and in which M is very large we find the minimum requirement on the flow velocity v for the liquid to be able to slow down:

$$v \geq E_{ex}/p_{ex} \quad (3)$$

In a normal liquid without an energy gap, liquid flow will always be damped because the minimum slope would be zero. This [Eq. 3] of course is also the reason why an ideal Bose gas does not become superfluid. Here the excitation energies are given by $E_{ex} = p_{ex}^2/2m$, and a parabolic curve does not have a minimum slope: no matter how slow an ideal Bose liquid is flowing, it is always possible to transfer energy by creating an excitation, and the liquid will slow down. Also, note that even though it requires less energy to create a sound wave than a roton excitation, the roton minimum

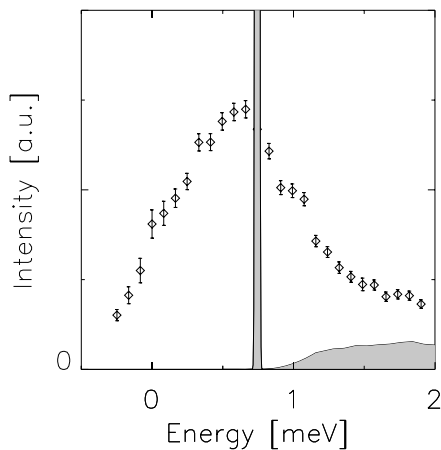


FIG. 3: Detailed view of the excitations of ^4He corresponding to the roton minimum in the superfluid phase (shaded) and in the normal fluid phase (points plus errorbars) [14]. The data are taken using neutron scattering. The neutron transfers energy E to the liquid and an amount of momentum corresponding to the roton minimum [see vertical arrow in Fig. 1]. When the amount of energy transferred exactly matches the energy difference between the ground state and the excited state, then a sharp resonance peak [at 0.743 meV] can be seen in the superfluid. Note that there is no signal below this peak. In the normal fluid the behavior is very different; even a small amount of energy is sufficient to excite the liquid, and there is a clear signal even at $E = 0$. The signal at $E < 0$ implies that the liquid gives up some of its energy to the neutron, which can happen if the liquid is not at zero Kelvin. The height of the peak of the sharp resonance is such that the size of the shaded area is the same as the area under the curve for the normal fluid.

actually determines the critical flow velocity.

In this paper we explain why there is an energy gap in the first place, how the size of this gap relates to the superfluid transition temperature, and how actual values for all parameters involved can be estimated. Feynman explained in a beautiful argument why this energy gap is the unavoidable consequence of the fact that ^4He atoms obey Bose statistics. We refer the reader to Feynman's 1955 account [8] and 1972 textbook [9] for details, but in a nutshell the argument is the following.

Assume that a certain configuration of the helium atoms represents the state with lowest energy, the ground state. The quantum mechanical wave function ϕ of this state depends on the positions of all atoms: $\phi(\vec{R}_1, \vec{R}_2, \dots, \vec{R}_N)$. The energy of this ground state consists of a kinetic energy term that depends on the gradient of the wave function $\sim |\nabla\phi|^2$ as well as a potential term $V|\phi|^2$. [The same holds for the wave function $\psi(\vec{R}_1, \vec{R}_2, \dots, \vec{R}_N)$ describing the excited state that is lowest in energy of all excited states]. The potential operator has terms $\sim 1/|\vec{R}_i - \vec{R}_j|^n$ which tell us that the force between the atoms is strongly repulsive

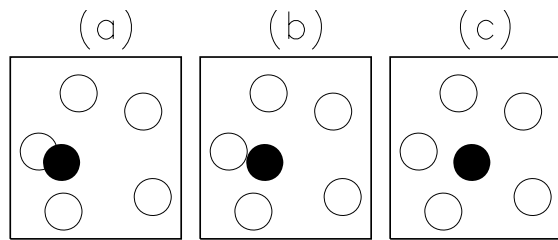


FIG. 4: A depiction of various configurations representing different energies [9]. Part (a) shows an unlikely ground state configuration since two atoms being in the same spot implies a high potential energy. Similarly, part (b) shows an unlikely ground state because this configuration would represent a high kinetic energy [see text]. On average, in the ground state the atoms will be spaced out as shown in part (c).

when they are too close together.

The exact details of the ground state are not important, but both the kinetic and potential term should be small. From this requirement we can expect that the atoms in a configuration that could represent the ground state are fairly well spread out [see Fig. 4c]. After all, if they were to sit on top of each other [Fig. 4a], we would pay a high price in potential energy, and hence, we can assume the amplitude of the ground state wave function ϕ to be zero for those cases where $R_i \approx R_j$. Also, the atoms will not be too close to each other [Fig. 4b], because this would correspond to a high gradient $\nabla\phi$, making it an unlikely choice of ground state. We can see that atoms almost touching each other would correspond to a high gradient as follows: if the amplitude of ϕ would not be zero for a configuration where two atoms are very close, then we would have the situation that by slightly changing the coordinate of one atom to make it sit on top of its neighbor [going from Fig. 4b to 4a], we would go from a non-zero to a zero amplitude for ϕ . This implies a steep gradient $\nabla\phi$, and therefore, the amplitude of ϕ must also be zero for configurations where atoms are too close. Another way of saying the above is that if an atom actually were to move from Fig. 4b to 4a, it must have had a large kinetic energy in the first place to be able to approach the other atom as closely as shown in Fig. 4a. However, note that we do not actually 'move' atoms, we just compare the wave function for two different configurations \vec{R}^N . Thus, when we say 'move', we do not imply any dynamics.

We also know that ϕ will not have any nodes other than at the edge of the box confining the liquid, so we can assume that the amplitude of ϕ is positive for all configurations. Also, the excited state wave function ψ should have at least one more node. This is essentially the guitar string equivalent of looking at the first and second harmonic. This implies that half of the configurations representing the excited state

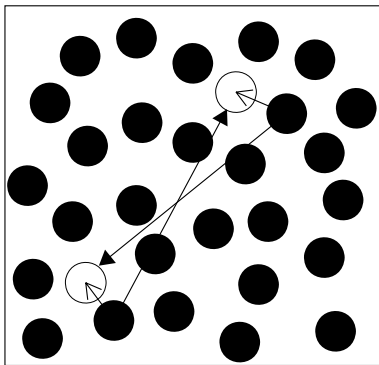


FIG. 5: A reasonable guess of what a low energy state in helium could look like [9]. To flip the sign of ψ atoms are 'moved' over large distances (long arrows) while smoothing out any holes left in the liquid to minimize the energy cost. However, since the Bose particles cannot be distinguished from each other and since permutation of particles does not affect the wave function, the outcome of the 'movements' indicated by the solid arrows are identical to the 'movements' indicated by the short arrows.

$\psi(\vec{R}_1, \vec{R}_2, \dots, \vec{R}_N)$ correspond to a positive amplitude, and half of the configurations correspond to a negative amplitude [10]. We will now try to create an excited state that barely differs in energy from the ground state. This way, the roton minimum would be very small [yielding a very small minimum slope] and we would be able to stifle superfluidity. However, Feynman showed that this cannot be done [8], and that one always ends up with a sizeable energy difference between the ground state and the excited states [that are not phonons].

In Fig. 5 we have sketched a configuration that we arbitrarily will take to correspond to a maximum positive amplitude for ψ . We do not really know what the configuration should look like, but we tried to make it look like the ground state, with atoms not sitting on top of each other. Next, we will rearrange the atoms to end up with a configuration that would correspond to a maximum negative amplitude. To achieve this, we should rearrange the helium atoms over large distances. We are not interested in short distances, because this would imply that ψ goes from a maximum to a minimum over short distances, which would correspond to a large gradient $\nabla\psi$ and therefore, to a high energy. We will also smooth out any holes or bumps that may materialize, otherwise we would end up with an excitation that looks like a phonon (see Fig. 2) and we already know that a phonon does not represent the minimum slope of the excitation curve. The required changes to the configuration are shown in Fig. 5 by the long arrows, and we appear to have achieved our aim.

However, the above approach does not work in a

Bose liquid since all the atoms are indistinguishable and interchanging two atoms does not lead to a change in the amplitude of the wave function. Thus, one could have gotten the same final configuration by simply 'moving' the atoms affected by the rearrangement over distances less than half the atomic separation [short arrows in Fig. 5]. In fact, half the atomic separation is the best that one could achieve. However, such a rapid variation (from maximum to minimum over half the atomic separation) would represent a large gradient and signify a significant step up in energy. In other words, because of the Bose nature of the atoms, it is not possible to make an excited state (which is not a phonon) that differs by a vanishingly small amount in energy from the ground state. Therefore, an energy gap must be present in a Bose liquid and provided the liquid does not freeze, it must become a superfluid. Whether a Bose-Einstein condensate forms or not is not relevant to this argument since it links the property of superfluidity to the scarcity of excited states. Even a Bose liquid where only a tiny fraction of the atoms condenses will become a superfluid when the temperature is low compared to the energy gap.

Thus, from a qualitative point of view, it is clear why a Bose liquid that remains liquid down to low enough temperatures has to become a superfluid because of the presence of an energy gap. However, this argument does not provide us with a numerical estimate for the size of the gap. Moreover, it does not even tell us how the transition temperature is linked to the size of a gap. This probably explains why textbooks tend not to mention Feynman's arguments.

SUPERFLUIDITY: QUANTITATIVE UNDERSTANDING

So how low does the temperature have to be for the liquid to become a superfluid? Since we do not think that the temperature at which BEC occurs in an ideal Bose gas made up of non-interacting atoms with the same mass as helium atoms [3.13 K] has much to do with the magnitude of the energy gap in a real liquid, we must take a different approach. We first discuss a relationship between the size of the energy gap and the superfluid transition temperature, followed by a discussion on how to estimate the size of the gap based on the speed of sound and the particle density of the liquid. Doing so, we will end up with an accurate estimate of the superfluid transition temperature. Our estimates apply to ^4He at zero pressure, possible extensions to higher pressures are given in footnotes.

As a note of caution, while our estimates turn out to be remarkably accurate [probably because of cancelation of errors], the following should not be read as anything

other than being a set of instructional exercises to help students in their understanding of liquid helium in particular, and in using their acquired knowledge from introductory physics to real world problems. It is not a derivation of the transition temperature in superfluid helium, even though our estimate turns out to be very accurate.

Connection T_λ to excitation gap

We first apply the same reasoning as in the preceding section to figure out what the minimum velocity requirement is for a single helium atom. For this atom of mass m moving with speed v through the sea of other atoms to be able to slow down to speed v' we have:

$$\begin{aligned} mv^2/2 &= mv'^2/2 + E_{ex}(p) \\ m\vec{v} &= m\vec{v}' + \vec{p}_{ex}. \end{aligned} \quad (4)$$

Since we are interested in the bare minimum, we assume that this atom will give up all of its energy [$v'=0$]. Dividing the above equations we get a minimum condition on the speed of the atom for it to be able to transfer energy to the rest of the liquid:

$$v_{minimum} \geq 2[E_{ex}/p_{ex}]_{minimum} \quad (5)$$

Compared to Eq. 3 we have picked up a factor of 2, which is the result of dealing with a small mass m instead of with the large mass M of a moving liquid. Note that the actual mass of the atom does not play a role in this expression. This ensures the validity of Eq. 5 in real liquids in which the actual movement of an atom is accompanied by a flow pattern where atoms are temporarily pushed out of the way, bestowing the moving atom with an effective mass which is larger than that of the mass of a non-interacting atom [about 2~3 times larger, see ref [11] for details].

If an atom moves faster than this minimum requirement it can create an excitation in the rest of the liquid and slow down, if it moves slower then it will not be able to slow down (that is, it will not experience any friction). Thus, we should expect to see a qualitative difference in behavior of the liquid above and below the temperature at which the thermal velocities meet the minimum requirement contained in Eq. 5. To estimate this temperature, we assume that the classical equipartition of energy principle can be extended to quantum liquids at low temperature, namely $mv_{thermal}^2/2 = 3k_B T/2$. Combining this with Eq. 5 we find that we can expect changes in liquid behavior at a temperature T_λ when $v_{thermal} = v_{minimum}$:

$$T_\lambda = \frac{4m[E_{ex}/p_{ex}]_{minimum}^2}{3k_B}. \quad (6)$$

We read off the value of the minimum slope [11] from Fig. 1: $(E_{ex}/p_{ex})_{minimum} = (58.05 \pm 0.15)$ m/s which combined with the mass of a helium atom of 6.646×10^{-27} kg yields a transition temperature of $T_\lambda = 2.162$ K ± 0.012 K, in good agreement with the actual transition temperature of 2.17 K. The fact that the agreement is essentially perfect might be fortuitous, however, it does show that our assumption of being able to use the equipartition of energy theorem to be not too far off the mark [12]. Also, Eq. 6 tells us that the transition temperature is determined by an intrinsic microscopic velocity of the liquid, as we would have expected for superfluidity.

Connection excitation gap to speed of sound

Now that we have made the connection between the macroscopic transition temperature and the microscopic parameters for the roton, we can try to estimate these roton parameters based on other macroscopic quantities. We start with the value of the energy gap at the roton minimum. As an aside, we note that the entire excitation curve shown in Fig. 1 can in principle be calculated with great precision from first principles [13], including the flat part at higher momentum values [$p^* \geq 3 \text{ \AA}^{-1}$] and its termination at twice the roton energy [14]. The procedure is straightforward, but cumbersome. Since we only need the value of the energy gap at the roton minimum, we use the following more instructive shortcut.

As Feynman pointed out in his argument why a Bose liquid should exhibit an energy gap [8], one gets the lowest lying excited state when one 'moves' atoms by half the atomic separation. We can use this to calculate the value of the energy gap based on the macroscopic speed of sound in superfluid helium. From thermodynamics we have that the speed of sound c is given by

$$c^2 = \frac{\gamma}{m} \left(\frac{\partial P}{\partial n} \right)_T \quad (7)$$

with P the pressure and γ the ratio of specific heat at constant pressure c_p and at constant volume c_v . For superfluid helium at low temperature we find that $\gamma = 1$ [16] as a direct consequence of the large zero-point motion of the atoms[17]. Thus, at constant volume V we have

$$c^2 = \frac{1}{m} \left(\frac{\partial PV}{\partial N} \right)_{T,V}. \quad (8)$$

We can now calculate c by adding one more atom to the liquid at constant volume. In this case, $\Delta N = 1$ and ΔPV is the amount of work we have to do to make room for this additional atom.

We calculate this amount of work by comparing the energy of a configuration with a hole in it to the energy of a configuration without such a hole. We can make a cubic hole in the liquid by 'moving' atoms along the positive x-direction by half the atomic separation $d/2$, and by doing the same thing along the negative x-direction, and by repeating the process in the y and z-directions. Each of these 6 configurational changes should increase the energy of our state, but only by E_{roton} for every step. This can be seen as follows: provided we can 'move' atoms over a distance of at least $d/2$, and provided we have plenty of room to smooth out any variations in local particle density, then we should be able to do each step of the process at a minimum energy cost. Thus the total cost will be six times the minimum excitation energy in liquid helium, or $6E_{\text{roton}}$. Of course in doing this, we actually made the hole too big since we only needed to make a sphere of diameter d . In all, we only need to provide $6E_{\text{roton}}[(4\pi/3)(d/2)^3]/d^3 = \pi E_{\text{roton}}$ in work. Combining all this we find [15]

$$E_{\text{roton}} = mc^2/\pi. \quad (9)$$

To see how reliable an estimate this is, we compare this prediction to the measured quantities of superfluid helium at 1.2 K. Using $c = 237.4$ m/s [16] (Fig. 1), we obtain $E_{\text{roton}} = 11.92 \times 10^{-23} \text{ J} = 0.744$ meV. The value that has actually been measured by means of neutron scattering [3] is 0.743 meV. Thus, we have found a very accurate value for the energy gap based on the speed of sound. In essence, we have used the speed of sound to gauge the strength of the interatomic potential, which in turn determines the roton energy.

Connection roton excitation to liquid density

So far we have connected the superfluid transition temperature to a minimum speed which can be determined from the excitation curve shown in Fig. 1, and we have connected the minimum excitation energy E_{roton} to the macroscopic speed of sound. To be complete, we should also estimate the momentum value corresponding to this minimum excitation energy, so that we can get an overall estimate of the minimum value of $E_{\text{ex}}/p_{\text{ex}}$ for the excitations shown in Fig. 1. We expect $E_{\text{roton}}/p_{\text{roton}}$ to be a very accurate estimate of this minimum, perhaps fractionally too large [see Fig. 1]. The roton momentum p_{roton} is given by $p_{\text{roton}} = h/\lambda_{\text{roton}} = h/d$, with h Planck's constant and d the interatomic separation. In words, the energy cost to create an excitation of wave length λ is least when this wave length matches the natural length scale in the liquid, the atomic separation.

The reader might be surprised to see that the roton wave length corresponds to d instead of $d/2$. This goes

back to our earlier usage of the verb to 'move'. We did not actually move a single atom over $d/2$, nor did we say that moving a single atom would correspond to the roton excitation. The only fact we used was that the entire configuration described by the excited state wave function ψ shown in Fig. 5 contained a roton excitation. The question we ask now is subtly different: if we were to characterize the excited state wave function by the average distance between atoms, what distance would we find? The answer is d since the excited state is similar to the ground state in many ways [in fact we constructed it so that the two would be as close as possible]. Whatever a roton excitation might be, it is some arrangement where the average separation between the atoms is pretty much as it is in the ground state, otherwise the energy of the excited state would be much higher.

The separation d depends on the number density n of liquid helium as $d \sim n^{-1/3}$. Estimating the proportionality factor is a somewhat nebulous undertaking because unlike in a solid, we do not have a nice periodic arrangement. For our estimate we use that helium fairly accurately resembles a liquid of closely packed spheres. The fraction of the volume occupied by the atoms in such a liquid is $\pi/3\sqrt{2} = 0.741$, so that we estimate the atomic separation to be [18]

$$d = [\pi/(3n\sqrt{2})]^{1/3}; p_{\text{roton}} = h[\pi/(3n\sqrt{2})]^{-1/3}. \quad (10)$$

This separation is slightly lower than that for a simple cubic structure [$d = (1/n)^{1/3}$] which simply tells us that the atoms are closer together in a closely packed structure. Combining Eqs. 6, 9 and 10 we get

$$T_\lambda = \frac{4m}{3k_B} \left[\frac{mc^2(\pi/3\sqrt{2})^{1/3}}{h\pi n^{1/3}} \right]^2. \quad (11)$$

When we plug in all the numbers [$c = 237.4 \pm 0.5$ m/s, $n = 0.02183$ atoms/ \AA^3], we find $T_\lambda = 2.17 \pm 0.02$ K. Thus, when we combine our quantitative approximation for the relationship between the minimum of the dispersion curve and T_λ [Eq. 5], with our approximation for the roton energy [Eq. 9], and with our approximation for the roton position [Eq. 10], we still find very good agreement with experiment. While it is satisfying to have ended up with such a good agreement, we note that the agreement is probably better than we had reason to expect given the simplicity of our estimates. From an instructional point of view however, we consider the individual links that we have made between microscopic parameters and macroscopic quantities [that is, Eqs. 5, 9 and 10] to be the most important points of this section since it allows students to apply basic physics reasoning in order to arrive at predictions for measurable quantities.

In summary, we have shown that superfluidity can be explained to students without going into lengthy

calculations, and without having to invoke a Bose condensate. We have tied Feynman's arguments about the origin of the energy gap to the actual superfluid transition temperature, and we have shown that very accurate estimates of all parameters involved can be obtained through straightforward reasoning. While we have included the actual numerical values for our calculations in this paper to make the discussion less abstract, and while we have even given an expression of the superfluid transition temperature in terms of macroscopic quantities like density and speed of sound, the real message of the paper is that students should be able to develop a better sense of what causes the property of superfluidity in terms of Bose statistics and energy gaps; that is, better compared to the standard Bose-Einstein condensation remarks that are normally encountered in textbooks. Finally, since a very similar relationship exists between the transition temperature and the energy gap in superconductors, this paper could also serve as an introduction to the physics of superconductivity.

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- [15] This relation [Eq. 9] can be generalized to cover higher pressures by including a term $\sim PV$.
- [16] J. Maynard, *Phys. Rev. B* **14**, 3868-3891 (1976).
- [17] The difference in specific heats is proportional to the thermal expansion coefficient α as $c_p - c_v \sim \alpha^2$. At low temperatures, $\alpha = 0$ because the density of helium is determined by the large zero-point-motion of the particles; reducing the temperature will not lead to any increase in density.
- [18] The inverse of this relation [$p_{roton}^* = 2\pi/d$] gives an accurate prediction for the position of the roton minimum for all pressures which is known to vary as $\sim n^{1/3}$, see ref. [16].